Peaks, Gaps and Time Reversibility of Economic Time Series

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SIdE Webinars 2021 series

May 7, 2021

Introduction

- Given a stochastic process Y_t, we consider two derived processes: the *max* process, which identifies the running maximum of a series and the *gap* process, which measures the deviation of the current value from the running maximum.
- The gap provides a measure of depth of a recession and of a bear market phase.
- In association with the running maximum, we define the process S_t, generated by the running time lag from the maximum. This is a discrete state first order Markov chain adapted to the natural filtration generated by Y_t.
- We derive the ergodic and transition probabilities as a function of the mean and the autocovariance function of the first differences of the process. Hence, we show that the gap process is covariance stationary and derive its moments.
- A new test of time reversibility of economic time series is proposed and applied.
- Additional related processes can aid the analysis of the business cycle, measuring the range of economic fluctuations, and the duration of bear and bull markets.

Max and Gap processes

Let $\{Y_t, t = 0, 1, \ldots, n\}$, denote a discrete–time continuous–state random process.

Let $\{\Delta_i Y_t, t = 1, \ldots, n\}$ denote the lag i difference process, $\Delta_i Y_t = Y_t - Y_{t-i}$, where we write $\Delta = \Delta_1$, so that $\Delta_i Y_t = \sum_{j=0}^{i-1} \Delta Y_{t-i}$.

The maximum process, M_t, and the associated gap process, G_t, are defined respectively as

$$M_t = \max\{Y_t, Y_{t-1}, \ldots, Y_{t-q}\}, \quad G_t = Y_t - M_t, \quad t = q, q+1, \ldots, n. \tag{1}$$

M_t defines a random lagged value of Y_t, whereas G_t is a change variable.

$m_t = y_{t-10}$

$g_t = y_t - y_{t-10}$

- The max and gap processes have several applications in economics and image processing.
- In economics the gap is also referred to as the *current depth of recession* (CDR). It was originally proposed by Beaudry and Koop (1993) for q equal to $t-1$; in particular, they defined $CDR_t = \max\{Y_{t-j}\}_{j\geq 0} - Y_t$.
- The variable $-CDR_t$ is used as an explanatory variable in the floors and ceilings models by Pesaran and Potter (1997), extended by Koop and Potter (2006) to the multivariate case.
- In their study of US unemployment, Parker and Rothman (1997) include (lagged values of) $\min_{j=0,\ldots,r} Y_{t-j} - Y_t$ as an explanatory variable in an ARMA framework, measuring the distance of the current rate from its historical local minimum.
- In signal and image processing the local max-filter, also referred to as a dilation filter, is used for demodulation of signals and denoising. The max-filter has interesting extension to rank order filters; see Barner and Arce (1998 and Maragos (2009)).

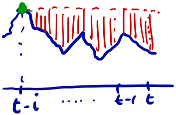
The Markov Chain generated by the Max Filter

Consider the associated process S_t with $q+1$ states

$$S_t = i \Leftrightarrow M_t = Y_{t-i}, \quad i = 0, \dots, q.$$

This is the (random) time lag with respect to the current maximum.

When $S_t = i$, $M_t = Y_{t-i}$ and $G_t = -|Y_t - Y_{t-i}|$.
This occurs with probability



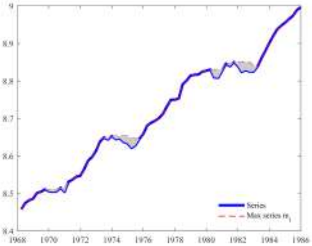
$Y_t < Y_{t-i}$
$Y_{t-1} < Y_{t-i}$
$\cdots$
$Y_{t-i+1} < Y_{t-i}$
$Y_{t-i-1} < Y_{t-i}$
$\vdots$
$Y_{t-q} < Y_{t-i}$

$$\begin{aligned}\pi_{it} &= P(S_t = i) \\ &= P(\Delta_i Y_t < 0, \dots, \Delta Y_{t-i+1} < 0, \Delta Y_{t-i} \geq 0, \dots, \Delta_{q-i} Y_{t-i} \geq 0).\end{aligned}$$

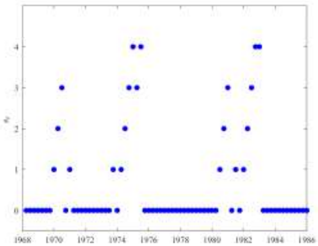
The event in parenthesis corresponds to the occurrence of a local maximum at time $t-i$.

Hereby we provide a description of the probability distribution of S_t, M_t, G_t.

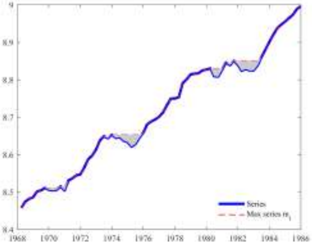
S_t	M_t	G_t	$\pi_{it} = P(S_t = i)$
0	Y_t	0	$P(\Delta Y_t \geq 0, \Delta_2 Y_t \geq 0, \cdots, \Delta_{q-1} Y_t \geq 0, \Delta_q Y_t \geq 0)$
1	Y_{t-1}	$-\lvert\Delta Y_t\rvert$	$P(\Delta Y_t < 0, \Delta Y_{t-1} \geq 0, \cdots, \Delta_{q-2} Y_{t-1} \geq 0, \Delta_{q-1} Y_{t-1} \geq 0)$
2	Y_{t-2}	$-\lvert\Delta_2 Y_t\rvert$	$P(\Delta_2 Y_t < 0, \Delta Y_{t-1} < 0, \Delta Y_{t-2} \geq 0, \cdots, \Delta_{q-2} Y_{t-2} \geq 0)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$
i	Y_{t-i}	$-\lvert\Delta_i Y_t\rvert$	$P(\Delta_i Y_t < 0, \ldots, \Delta Y_{t-i+1} < 0, \Delta Y_{t-i} \geq 0, \ldots, \Delta_{q-i} Y_{t-i} \geq 0)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$
$q-1$	Y_{t-q+1}	$-\lvert\Delta_{q-1} Y_t\rvert$	$P(\Delta_{q-1} Y_t < 0, \Delta_{q-2} Y_{t-1} < 0, \cdots, \Delta Y_{t-q+1} < 0, \Delta Y_{t-q} \geq 0)$
q	Y_{t-q}	$-\lvert\Delta_q Y_t\rvert$	$P(\Delta_q Y_t < 0, \Delta_{q-1} Y_{t-1} < 0, \cdots, \Delta Y_{t-q+1} < 0)$



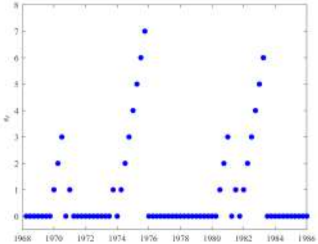
(a) Plot of y_t and m_t $(q = 4)$.



(b) States s_t $(q = 4)$.



(c) Plot of y_t and m_t $(q = 8)$.



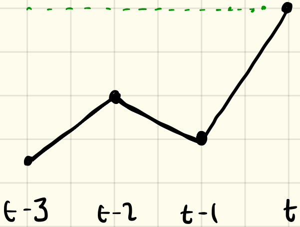
(d) States s_t $(q = 8)$.

- The properties of the variables S_t, M_t and G_t depend on the vector of q first differences
$$\Delta Y_{t,q} = (\Delta Y_t, \ldots, \Delta Y_{t-q+1})'.$$

- If ΔY_t is a stationary process, then there exist a stationary distribution $\pi_{it} = \pi_i, i = 0, \ldots, q$, for all $t \geq q$.

- In the paper we derive closed form formulae for the unconditional (ergodic) probabilities $\pi_i = P(S_t = i)$, $i = 0, \ldots, q$.

$q = 3$

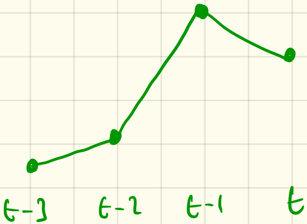


$S_t = 0$

$$\pi_0 = P(S_t = 0) = P(\Delta y_t > 0, \Delta_2 y_t > 0, \Delta_3 y_t > 0)$$

$$= P(W_0 \Delta Y_{t,3} > 0)$$

$$W_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \Delta Y_{t,3} = \begin{pmatrix} \Delta y_t \\ \Delta y_{t-1} \\ \Delta y_{t-2} \end{pmatrix}$$



$S_t = 1$

$$\pi_1 = P(S_t = 1) = P(\Delta y_t < 0, \Delta y_{t-1} > 0, \Delta_2 y_{t-1} > 0)$$

$$= P(W_1 \Delta Y_{t,3} > 0)$$

$$W_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \Delta Y_{t,3} = \begin{pmatrix} \Delta y_t \\ \Delta y_{t-1} \\ \Delta y_{t-2} \end{pmatrix}$$

Similarly,

$$\pi_2 = P(S_t = 2) = P(\Delta_2 y_t < 0, \Delta y_{t-1} < 0, \Delta y_{t-2} > 0)$$

$$= P(W_2 \Delta Y_{t,3} > 0) \qquad W_2 = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\pi_3 = P(S_t = 3) = P(W_3 \Delta Y_{t,3} > 0) \qquad W_3 = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

Transition probabilities

- We henceforth assume that ΔY_t is a stationary process (in a strong sense).
- The process S_t is a first order Markov chain with transition probabilities

$$p_{ij} = P(S_{t+1} = j | S_t = i), i, j = 0, 1, \dots, q,$$

  which can be obtained uniquely from the joint distribution of the $q + 1$ consecutive differences $\{\Delta Y_{t+1-k}, k = 0, \dots, q\}$.
- From state i, $i = 0, \dots, q - 1$, the only admissible transitions are to 0 or $i + 1$. From state q a transition can be made to any other state.

The transition probabilities are collected in the following upper Hessenberg transition matrix:

$$T = \begin{pmatrix} p_{00} & p_{01} & 0 & \dots & 0 & 0 \\ p_{10} & 0 & p_{12} & \dots & 0 & 0 \\ p_{20} & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots \\ p_{q-1,0} & 0 & 0 & \cdots & 0 & p_{q-1,q} \\ p_{q0} & p_{q1} & p_{q2} & \cdots & p_{q,q-1} & p_{qq} \end{pmatrix}. \tag{2}$$

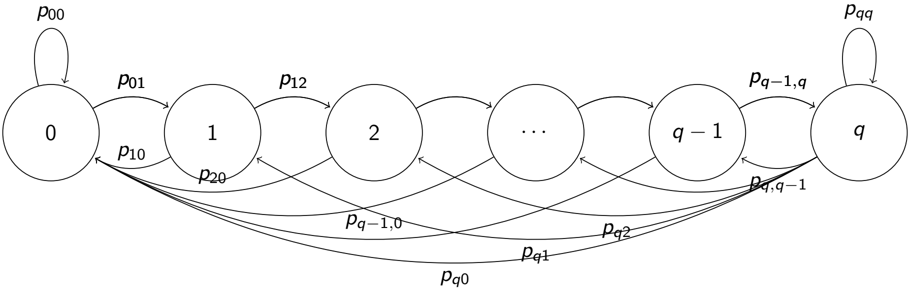
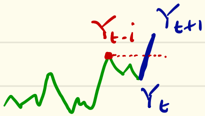


Figure: State transition diagram of S_t



$$p_{io} = P(S_{t+1} = 0 \mid S_t = i) = \frac{P(S_{t+1} = 0,\ S_t = i)}{P(S_t = i)}$$

$$= \frac{P(\Delta_{i+1} Y_{t+1} > 0,\ W_i \Delta Y_{t,p} > 0)}{P(W_i \Delta Y_{t,p} > 0)}$$

$$= \frac{1}{\pi_i} P(W_{io} \Delta Y_{t+1,q+1} > 0)$$

where

$$W_{io} = \left[\begin{array}{c|c} 1 & 1 \cdots 1\ 0 \cdots 0 \\ \hline 0 & \\ \vdots & W_i \\ 0 & \end{array}\right] \qquad \Delta Y_{t+1,q+1} = \begin{bmatrix} \Delta y_{t+1} \\ \Delta y_t \\ \vdots \\ \Delta y_{t-q+1} \end{bmatrix}$$

Weak stationarity of the gap process

- If ΔY_t is a stationary process, G_t is weakly stationary.
- In the paper we derive the unconditional mean, variance and autocovariance function of the gap process.
- Closed form expression are available if Y_t is Gaussian process, by using the moments of the multivariate folded normal distribution.

Example: Gaussian ARIMA(1, 1, 0) process

Let us consider the process $\Delta Y_t - \mu = 0.8(\Delta Y_{t-1} - \mu) + \epsilon_t, \epsilon_t \sim$ i.i.d. $N(0, \sigma^2)$ and $q = 5$.

	Values of μ/σ						
	-2.33	-1.64	-0.67	0.00	0.67	1.64	2.33
π_0	0.00	0.01	0.14	0.38	0.67	0.93	0.99
π_1	0.00	0.01	0.04	0.07	0.06	0.02	0.01
π_2	0.00	0.01	0.04	0.06	0.05	0.01	0.00
π_3	0.00	0.01	0.05	0.06	0.04	0.01	0.00
π_4	0.01	0.02	0.06	0.07	0.04	0.01	0.00
π_5	0.99	0.93	0.67	0.38	0.14	0.01	0.00
p_{00}	0.50	0.60	0.74	0.83	0.91	0.98	0.99
p_{qq}	0.99	0.98	0.91	0.83	0.74	0.60	0.50
$E(G_t)$	-11.57	-8.01	-3.41	-1.73	-1.03	-0.31	-0.07
$Var(G_t)$	18.27	17.64	10.08	5.34	4.73	2.19	0.56
ACF(1)	0.96	0.94	0.89	0.86	0.84	0.79	0.72
ACF(5)	0.49	0.45	0.32	0.24	0.26	0.18	0.10
ACF(10)	0.16	0.14	0.08	0.05	0.06	0.02	0.01

Inference

- A sample of size n is available.
- Define the indicator function $I_{it} = I(S_t = i)$, taking value 1 if $S_t = i$ and zero otherwise, and the vector $I_t = [I_{0t}, \dots, I_{it}, \dots, I_{qt}]'$
- Letting $n^* = n - q$,

$$n_{ij} = \sum_{t=q+1}^{n} I_{i,t-1} I_{jt}, \quad n_{i.} = \sum_{t=q+1}^{n} I_{i,t-1},$$

$$\hat{\pi} = \frac{1}{n^*} \sum_{t=q}^{n} I_t, \qquad \hat{p}_{ij} = \frac{n_{ij}}{n_{i.}} \tag{3}$$

- Also define the joint probabilities $f_{ij} = \pi_i p_{ij}$, and $\hat{f}_{ij} = \frac{n_{ij}}{n^*-1}$.
- Define the set of admissible indices $\mathcal{A} = \{(i,j) : (\{0, \dots, q\}, j = 0) \cup (q, \{1, \dots, q\})\}$

Theorem

Let $\{\Delta Y_t, t \in \mathbb{Z}\}$ be strictly stationary and α-mixing of size $-r/(r-2)$, $r > 2$. Assume that the associated Markov chain $\{S_t, t \in \mathbb{Z}\}$ is such that $0 < \pi_i < 1$, $i = 0, \ldots, q$, and the transition probabilities p_{ij} satisfy $0 < p_{ij} < 1$, for $(i,j) \in \mathcal{A}$ and that $Var(\sum_t I_{it}) > 0$, $Var(\sum_t I_{i,t-1} I_{jt}) > 0$.
Then,

$$\sqrt{n^*}(\hat{\pi} - \pi) \rightarrow_d N(0, \Omega),$$

$$\Omega = (D_\pi - \pi\pi') + \sum_{k=1}^{\infty} \left\{ \left(T^k - 1_{q+1}\pi' \right) \odot \pi 1'_{q+1} + \left(T^{k'} - \pi 1'_{q+1} \right) \odot 1_{q+1}\pi' \right\}, \tag{4}$$

where $D_\pi = diag(\pi_0, \pi_1, \ldots, \pi_q)$.
For $(i,j), (h,l) \in \mathcal{A}$ and $(h,l) \neq (i,j)$

$$\sqrt{n^*} \begin{pmatrix} \hat{f}_{ij} - f_{ij} \\ \hat{f}_{hl} - f_{hl} \end{pmatrix} \rightarrow_d N \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{(ij)} & \sigma_{(ij)(hl)} \\ \sigma_{(ij)(hl)} & \sigma_{(hl)} \end{pmatrix} \right\},$$

with elements

$$\begin{aligned} \sigma_{(ij)} &= f_{ij}(1 - f_{ij}) + 2\textstyle\sum_{k=1}^{\infty} f_{ij}(p_{ij} p_{ji}^{(k-1)} - f_{ij}), \\ \sigma_{(ij)(hl)} &= \textstyle\sum_{k=1}^{\infty} \left\{ f_{hl}(p_{ij} p_{ji}^{(k-1)} - f_{ij}) + f_{ij}(p_{hl} p_{lh}^{(k-1)} - f_{hl}) \right\}. \end{aligned}$$

Time reversibility

The classical definition of time-reversibility (Tong, 1990), deals with a stationary process Y_t, for which, for any positive integer n, and every collection of time points $\{t_i \in \mathbb{Z}, i = 1, \dots, n\}$, the vectors $(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n})'$ and $(Y_{-t_1}, Y_{-t_2}, \dots, Y_{-t_n})'$ have the same joint probability distribution.

Tests of reversibility that consider the equality of the bivariate distributions (Y_t, Y_{t-k}) and (Y_t, Y_{t+k}) have been based proposed by Darolles et al (2004), Beare and Seo (2014). Ramsey and Rothman (1996) proposed a test of reversibility based on the contrast between the bicovariances $\mathrm{E}(Y_t^2 Y_{t-k})$ and $\mathrm{E}(Y_t Y_{t-k}^2)$, $k = 1, 2, \dots$, and Hinich and Rothman (1998) proposed a test in the frequency domain based on the bispectrum.

The literature has considered tests *lag-reversibility*, Lawrance (1991): if Y_t is stationary and time reversible, then the marginal distribution of $\Delta_i Y_t$ is symmetric about zero for all $i \in \mathbb{Z}$. Chen et al. (2000, CCK) proposed a test of symmetry based on the characteristic function. The sign test statistic by Psadarakis (2008) provides a nonparametric test of $P(\Delta_i Y_t > 0) = 0.5$. Racine and Maasoumi (2007) considered entropy based tests.

Tests of conditional symmetry are proposed by Chen and Kuan (2002), Bai and Ng (2001) and Delgado and Escanciano (2007), among others.

Definition (Local reversibility)

The stochastic process $\{Y_t, t \in \mathbb{Z}\}$ is said to be locally time reversible at time u with horizon q, if for a fixed $n \leq q$ and every collection of time points $\{t_i \in \mathbb{N}, 1 \leq t_1 < t_2 < \cdots < t_n \leq q\}$,

$$P(Y_{u+t_1} < y_1, Y_{u+t_2} < y_2, \ldots, Y_{u+t_n} < y_n | Y_u) = P(Y_{u-t_1} < y_1, Y_{u-t_2} < y_2, \ldots, Y_{u-t_n} < y_n | Y_u). \quad (5)$$

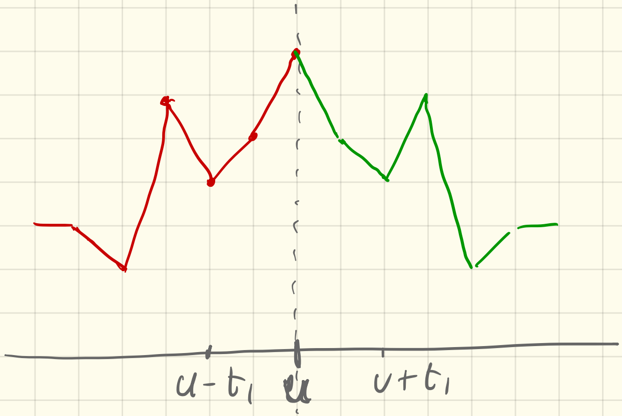
The definition does not requires strict stationarity of $\{Y_t\}$, nor it implies it. Invariance of (??) with respect to the choice of time u leads to the following definition of global reversibility.

Definition (Global reversibility)

The stochastic process $\{Y_t, t \in \mathbb{Z}\}$ is said to be globally time reversible with horizon q if it is locally reversible at every time $u \in \mathbb{Z}$ and

$$P(Y_{u+t_1} < y_1, Y_{u+t_2} < y_2, \ldots, Y_{u+t_n} < y_n | Y_u) = P(Y_{t_1} < y_1, Y_{t_2} < y_2, \ldots, Y_{t_n} < y_n | Y_0). \quad (6)$$

Definition 2 implies that the process is difference stationary. If $q \to \infty$ and Y_t is strictly stationary, we obtain the standard definition of time reversibility.



$\Delta Y_{t,q}$ is centrally symmetric

The joint cdf of $\Delta Y_{t,q}$ is equal to that of $-\Delta Y_{t,q}$

Testing for symmetry and time reversibility

- For a globally time-reversible process $\Delta Y_{t,q}$ is *centrally symmetric* about zero (Serfling, 2014), i.e., it has the same distribution as $-\Delta Y_{t,q}$, for every $q > 0$.
- Hence, S_t has the following symmetry properties:

$$\begin{aligned} \pi_i &= \pi_{q-i}, \quad i = 0, \ldots, \lfloor q/2 \rfloor, \\ p_{00} &= p_{qq}. \end{aligned} \tag{7}$$

- The test statistic that we propose for $H_0 : p_{00} = p_{qq}$ is

$$t_{n,q} = \frac{n_{00} - n_{qq}}{\sqrt{\mathrm{Var}(n_{00} - n_{qq})}} \tag{8}$$

  It provides an asymptotically standard normal test.
- We can extend the definition of symmetry and time-reversibility to account for the presence of a drift. In particular, we say that $\Delta Y_{t,q}$ is centrally symmetric about the $\mu = \mathrm{E}(\Delta Y_t)$, if the distribution of the vector $\Delta Y_{t,q} - i_{q+1}\mu$ is the same as that of $i_{q+1}\mu - \Delta Y_{t,q}$.

Simulation experiment

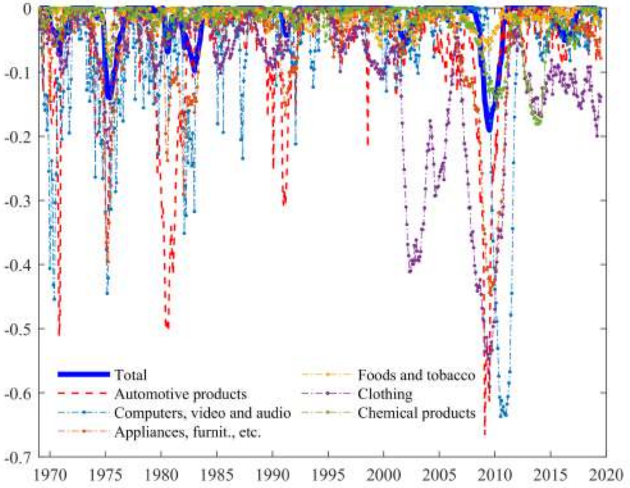
- The finite-sample performance of the proposed tests was evaluated by means of a simulation experiment.
- The results are reported in the paper.
- In the sequel we focus on $t_{n,q}$

Illustrations

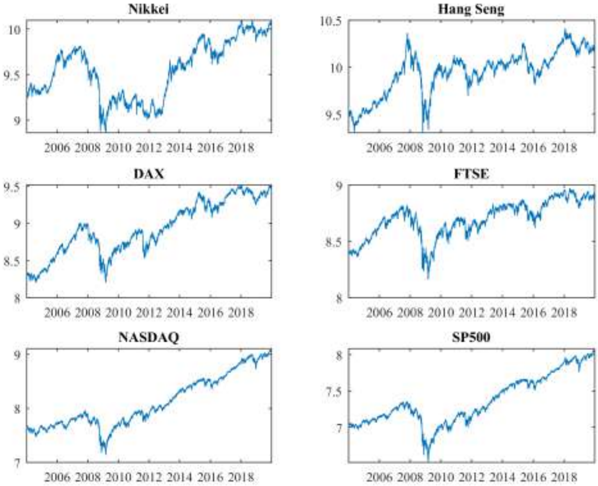
US Index of industrial production: values of the $t_{n,q}$ statistic for symmetry about 0 (top panel) and about the drift μ (bottom panel).

	Total	Auto. prod.	Comp., vid., aud.	Appl. furniture	Foods & tobacco	Clothing	Chemical products
$q = 1$	6.5360	1.7545	3.9164	1.7069	3.0662	-3.8284	3.9475
$q = 2$	5.7095	1.6641	4.3680	1.9382	3.6252	-3.9063	4.2914
$q = 6$	4.4850	1.8177	4.2255	1.3767	3.1172	-2.8590	3.0476
$q = 12$	6.3643	1.7457	3.5557	1.1188	2.9771	-2.4659	2.3804
$q = 24$	9.1059	1.6946	4.2604	0.9305	2.8565	-2.9489	1.8974
$q = 36$	4.8404	1.5230	3.1079	0.6928	1.9309	-1.9061	1.6344
$\sqrt{n}\hat{\mu}/\hat{\sigma}$	6.0734	1.3295	3.3613	0.8702	2.6172	-4.8344	4.0405
skewness	-1.0961	0.6624	0.0238	-0.5268	-0.0455	-0.3059	-0.2084
kurtosis	8.2468	17.3833	6.2150	6.7636	3.2728	4.6579	5.0647
			Test for symmetry of $Y_t - Y_0 - \mu t$				
$q = 1$	1.1025	0.8997	-0.0883	0.7238	-0.2702	1.7718	-0.3868
$q = 2$	0.7942	0.6392	-0.0922	0.9128	0.5082	1.3631	-0.6053
$q = 6$	0.8473	0.3438	0.4839	0.4263	0.2603	1.1104	-0.9765
$q = 12$	0.5686	-0.1102	0.0000	0.2157	-0.4551	1.1045	-0.9182
$q = 24$	0.3809	-0.1832	-0.2314	-0.0411	-0.6708	0.9849	-0.9141
$q = 36$	0.1869	-0.1551	-0.1746	-0.2215	-0.5387	1.0869	-0.6650

The output gap in industrial production



Six stock market indices (logs)



Values of the $t_{n,q}$ statistic for symmetry about 0 (top panel) and about the drift μ (bottom panel).

Sample 2004/01/01–2019/12/31

	Nikkei	Hang Seng	Dax	Ftse	Nasdaq	S&P500
$q = 1$	3.60	3.63	5.87	3.29	7.48	6.58
$q = 2$	3.50	2.88	5.38	3.68	7.98	6.14
$q = 5$	4.02	2.83	5.56	4.63	7.11	6.02
$q = 22$	3.70	2.42	4.89	3.58	5.84	5.26
$\sqrt{n}\hat{\mu}/\hat{\sigma}$	0.87	0.88	1.46	0.75	1.87	1.48
Skewness	-0.54	-0.02	-0.08	-0.17	-0.29	-0.38
Kurtosis	11.40	12.76	9.55	11.86	10.31	15.13

Sample 2004/01/01–2019/12/31, Mean corrected

	Nikkei	Hang Seng	Dax	Ftse	Nasdaq	S&P500
$q = 1$	2.08	2.41	4.07	2.32	4.54	4.31
$q = 2$	2.21	1.56	3.45	2.53	4.37	3.93
$q = 5$	2.56	1.68	3.73	3.68	4.22	3.83
$q = 22$	2.45	1.36	2.88	2.63	3.30	3.37

Values of the $t_{n,q}$ statistic for symmetry about 0 (top panel) and about the drift μ (bottom panel).

	Nikkei	Hang Seng	Dax	Ftse	Nasdaq	S&P500
Sample 2010/01/01–2019/12/31						
$q = 1$	3.70	2.17	4.04	2.34	6.89	5.29
$q = 2$	3.84	1.61	3.71	2.65	7.40	5.21
$q = 5$	4.18	1.51	3.79	3.35	7.24	5.42
$q = 22$	3.41	1.41	3.28	2.57	5.33	4.44
$\sqrt{n}\hat{\mu}/\hat{\sigma}$	1.23	0.45	1.31	0.69	2.51	2.25
Skewness	-0.57	-0.31	-0.31	-0.20	-0.45	-0.50
Kurtosis	8.53	5.29	5.76	5.51	6.42	7.59
Sample 2010/01/01–2019/12/31, Mean corrected						
$q = 1$	1.89	1.52	2.75	1.71	2.88	1.45
$q = 2$	2.15	1.12	2.21	1.91	2.89	1.70
$q = 5$	2.34	0.99	2.38	2.61	3.49	1.86
$q = 22$	1.91	1.04	1.78	1.84	2.07	1.75

Derived processes

- The minimum process, $M_t^- = \min\{Y_{t-i}, i = 0, 1, \ldots, q\}$, is obtained from the max-filter as $M_t^- = -\max\{-Y_{t-i}, i = 0, 1, \ldots, q\}$.
- The associated Markov chain, S_t^-, is in state i if the local minimum occurs at time $t - i$.
- We can also consider the gap process $G_t^- = Y_t - M_t^-$ and derive its properties.
- The *range* process is

$$\begin{aligned} R_t &= M_t - M_t^- \\ &= \max\{Y_{t-i} - Y_{t-j}, i, j = 0, 1, \ldots, q\} \\ &= G_t^- - G_t, \end{aligned} \tag{9}$$

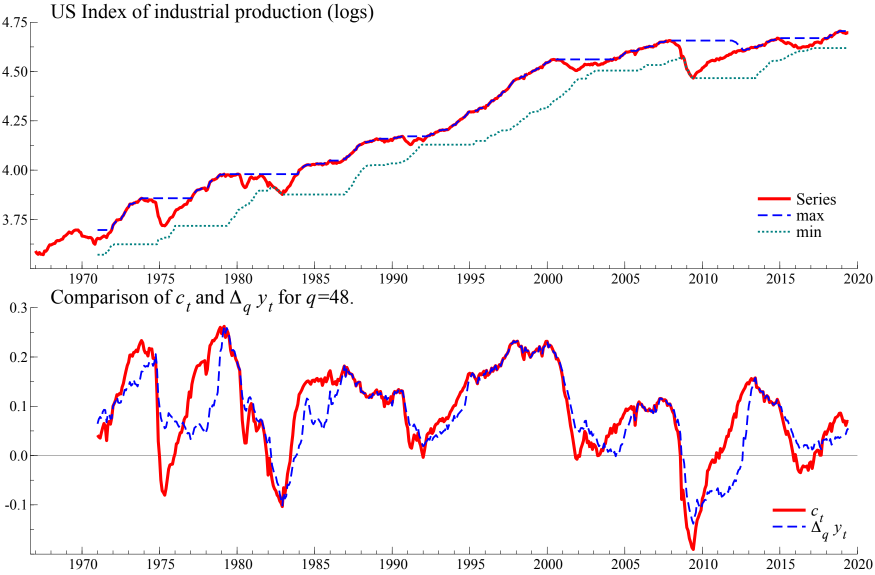
- R_t is covariance stationary if Y_t is either stationary or difference stationary.

- Other important associated processes are the *level* and *change* processes

$$T_t = \frac{1}{2}(M_t + M_t^-) = M_t^- + 0.5R_t, \quad C_t = G_t + G_t^- = 2(Y_t - T_t). \quad (10)$$

- C_t is related to the differences of the process. In particular, in times when Y_t behaves monotonically, $C_t = \Delta_q Y_t$, otherwise, if the maximum and minimum occur at times $t-i$ and $t-j$, respectively, $i \neq j$, $C_t = Y_t - Y_{t-i} + Y_t - Y_{t-j}$. This is exactly zero if Y_t lies in the middle between the historical maximum and minimum. At a local peak (trough), it equals $\Delta_j Y_t$ ($\Delta_i Y_t$), which is larger (smaller) than $\Delta_q Y_t$ if i (j) is less than q.
- As a result, the separation of the phases of the business cycle is more sharp.

US index of industrial production (total manufacturing). Min, max and change processes for $q = 48$ (4 years).



Recession duration

Another constructed variable deals with the duration of recessions or bear markets. Let us define the random variable D taking values $D = k$ if, after being in a peak at time t, the process lies below Y_t in the subsequent k times and peaks again at time $t + k + 1$. Hence, the recession has lasted k periods. We posit

$$\begin{aligned} P(D = k) &= P(S_{t+k+1} = 0, S_{t+k} = k, \ldots, S_{t+1} = 1 | S_t = 0) \\ &= \begin{cases} p_{00}, & k = 0, \\ p_{01} p_{12} \cdots p_{k-1,k} p_{k0}, & 1 \leq k \leq q. \end{cases} \end{aligned} \tag{11}$$

For values greater than q one can only provide $P(D > q) = 1 - \sum_{k=0}^{q} P(D = k)$. In a similar way, we can construct a variable for the duration of expansions or bull markets, D^*, taking the value k if, after a trough at time t (i.e., $S_t^- = 0$), $Y_{t+j}, j = 1, \ldots, k$, lies above Y_t and a new local minimum is reached at time $t + k + 1$.

SP500 index 2004-01-01 to 2019-31-12 ($n = 4026$).
Probability distribution of the duration of bear (D) and bull (D^*) market.

k	$P(D = k)$	$P(D > k)$	$P(D^* = k)$	$P(D^* > k)$
0	0.5084	0.4916	0.4102	0.5898
1	0.1454	0.3462	0.1523	0.4375
2	0.0881	0.2581	0.0602	0.3773
3	0.0370	0.2210	0.0371	0.3402
4	0.0360	0.1850	0.0298	0.3103
5	0.0213	0.1636	0.0139	0.2965
14	0.0043	0.0822	0.0036	0.2161
22	0.0042	0.0602	0.0065	0.1848

Conclusions

The analysis of the maximum and gap processes, the associated Markov chain, and other derived processes, adds useful insight into the properties of the random process Y_t for output and prices. In particular, it can be used for characterizing the depth and duration of a business cycle, or bear market, phase, and the time reversibility of economic fluctuations.

An interesting extension is dealing with the case when the underlying process Y_t is affected by measurement error. Suppose that we observe $X_t = Y_t + U_t$, where the additive error term, U_t, is a stationary random process with zero mean and possibly weak identifiable serial correlation structure. When Y_t and U_t can be given a linear Gaussian representation, the conditional distribution of $\{Y_t, t = 1, \ldots, n\}$, given $\{X_t = x_t, t = 1, \ldots, n\}$ is Gaussian and one can think of suitable simulation based inferences for the gap and related processes. We leave this extension to future research.